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Sufficient Regularity Conditions for Common Transversals

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A well-known condition sufficient for the existence of a transversal of a family of sets is generalized to common transversals of two families, in both the finite and the infinite cases.

1. INTRODUCTION

One of the very basic theorems in combinatorics is that of P. Hall [8], giving necessary and sufficient conditions for a finite family of sets to have a transversal. Over the years this theorem has spawned a multitude of others on the same general theme. In fact, a large enough body of mathematics has grown from it to form an entire area of combinatorics known as transversal theory. This area is intimately connected with matroid theory (see, e.g. [3]), although we will not pursue that connection below.

One major concern of transversal theory is to determine existence conditions for transversals and related structures. Most existing theorems of this type give necessary and sufficient conditions, usually showing that the existence of the desired structure is equivalent to the simultaneous truth of a large number of independent cardinality conditions (e.g., the P. Hall Theorem). The number of such conditions grows exponentially with the number of sets involved and, hence, the conditions are of limited value in a practical sense. For this reason, there is value to theorems which give more easily verifiable conditions sufficient for the existence of the structure in question. In this paper we take such a (known) condition for the existence of a transversal and generalize it to the question of the existence of a common transversal of two families of sets, in both the finite and infinite cases. Along the way we get a generalization of a well-known theorem of König.

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2. Fundamental Results for One Family of Sets

Let $\mathscr{S} = (S_i)_{i \in I}$ be a family of subsets of a set E. A set $K \subseteq E$ is a transversal of the family $\mathscr{S}$ if there exists a bijection $\rho : K \to I$ such that $k \in S_{\rho(k)}$ for all $k \in K$. Note that $\mathscr{S}$ may contain the same set more than once; hence the notation "$\mathscr{S} = (S_i)_{i \in I}$" rather than "$\mathscr{S} = \{S_i\}_{i \in I}$."

We begin by stating the above-mentioned Hall Theorem:

THEOREM 1 (P. Hall [8]). *A finite family $\mathscr{S} = (S_1, \ldots, S_n)$ of subsets of a set E has a transversal if and only if*

$$\left| \bigcup_{i \in I} S_i \right| \geqslant |I|$$

for all $I \subseteq \{1, \ldots, n\}$. (Throughout, "$|X|$" will denote the cardinality of the set X.)

We also state the generalization of Theorem 1 for infinite families:

THEOREM 2 (M. Hall [7]). *A family $\mathscr{S} = (S_i)_{i \in I}$ of finite subsets of a set E has a transversal if and only if*

$$\left| \bigcup_{i \in I'} S_i \right| \geqslant |I'|$$

for all finite $I' \subseteq I$.

Note that Theorem 2 does not go through if any of the sets S_i are allowed to be infinite; e.g.,

$$\mathscr{S} = (\{1, 2, 3, \ldots\}, \{1\}, \{2\}, \{3\}, \ldots)$$

has no transversal even though it satisfies the condition of Theorem 2 [7]. For infinite families of infinite sets very little is known. For what there is, see [1], [4], and [12].

There is a well-known corollary to Theorem 1 giving some regularity conditions which are sufficient for a transversal to exist.

Let $\mathscr{S} = (S_i)_{i \in I}$ be a family of subsets of a set $E = \{e_j\}_{j \in J}$. We say $\mathscr{S}$ is *(k, k')-regular* if $|S_i| = k$ $(0 < k < \infty)$ for all $i \in I$ and $|\{i \mid e_j \in S_i\}| = k'$ $(0 < k' < \infty)$ for all $j \in J$.

COROLLARY 3. *Let $\mathscr{S} = (S_1, \ldots, S_n)$ be a finite family of subsets of a set E, with $|E| \geqslant n$. If $\mathscr{S}$ is (k, k')-regular, then $\mathscr{S}$ has a transversal.*

Proof. Note that

$$\sum_{i=1}^{n} | S_i | = \sum_{e \in E} |\{i \mid e \in S_i\}|;$$

i.e., that

$$kn = k' \cdot | E | ,$$

and hence

$$\frac{k}{k'} = \frac{| E |}{n} \geqslant 1.$$

Now consider $| \bigcup_{i \in I} S_i |$ for any $I \subseteq \{1,..., n\}$. Since each set S_i has k elements, and since no element occurs more than k' times in these sets, we have

$$\left| \bigcup_{i \in I} S_i \right| \geqslant \frac{k \cdot | I |}{k'} \geqslant | I |.$$

Hence, $\mathscr{S}$ has a transversal by Theorem 1.

Corollary 3 is equivalent to a well-known graph-theoretic result of König [10, XI, Satz 13] which says that a regular bipartite graph has a complete matching. (This theorem corresponds directly to the case $| E | = n$ in Corollary 3, but can easily be extended to the more general situation.)

Corollary 3 can be restated in terms of 0-1 matrices by considering the incidence matrix of the family S. Call a set of entries in a 0-1 matrix *independent* if no two are in the same row or column. Then we have

COROLLARY 3′. *An $n \times m$ 0-1 matrix ($m \geqslant n$) with constant row sums (> 0) and constant column sums (> 0) contains a set of n independent 1's.*

The same line of proof as in Corollary 3 gives the corresponding corollary to Theorem 2:

COROLLARY 4. *Let $\mathscr{S} = (S_i)_{i \in I}$ be a family of finite subsets of a set E. If $\mathscr{S}$ is (k, k')-regular with $k \geqslant k'$, then $\mathscr{S}$ has a transversal.*

Note that the proof of Corollary 3 did not really use the full strength of the regularity condition. We actually proved something stronger:

COROLLARY 5. *Let $\mathscr{S} = (S_1, \ldots, S_n)$ be a finite family of subsets of a set E. If*

$$\min_{1 \leqslant i \leqslant n} |S_i| \geqslant \max_{e \in E} |\{i \mid e \in S_i\}|,$$

then $\mathscr{S}$ has a transversal. (This corollary also extends in the obvious way to the infinite case of Theorem 2.)

3. RESULTS FOR TWO FINITE FAMILIES

Let $\mathscr{S} = (S_i)_{i \in I}$ and $\mathscr{T} = (T_j)_{j \in J}$ be families of subsets of a set E. A set $K \subseteq E$ is a *common transversal of* the families $\mathscr{S}$ and $\mathscr{T}$ if it is a transversal of each family individually.

The fundamental theorem here is due to Ford and Fulkerson [6, II.10]:

THEOREM 6. *Two finite families $\mathscr{S} = (S_1, \ldots, S_n)$ and $\mathscr{T} = (T_1, \ldots, T_n)$ of subsets of a set E have a common transversal if and only if*

$$\left| \bigcup_{i \in J} S_i \cap \bigcup_{j \in J} T_j \right| \geqslant |I| + |J| - n$$

for all $I, J \subseteq \{1, \ldots, n\}$.

We now extend the regularity condition of Corollary 3 to the case of common transversals:

THEOREM 7. *Let $\mathscr{S} = (S_1, \ldots, S_n)$ and $\mathscr{T} = (T_1, \ldots, T_n)$ be two finite families of subsets of a set E, with $|E| \geqslant n$. If $\mathscr{S}$ is (k, k')-regular and $\mathscr{T}$ is (l, l')-regular, then $\mathscr{S}$ and $\mathscr{T}$ have a common transversal.*

Proof. Let $I, J \subseteq \{1, \ldots, n\}$, and let $|E| = m$ (the regularity of $\mathscr{S}$ and $\mathscr{T}$ implies that E is finite). Then, since $|A \cap B| = |A| + |B| - |A \cup B|$, we have

$$\left| \bigcup_{i \in I} S_i \cap \bigcup_{j \in J} T_j \right| \geqslant \frac{k \cdot |I|}{k'} + \frac{l \cdot |J|}{l'} - m.$$

But $k/k' = l/l' = m/n$, so we have

$$\left| \bigcup_{i \in I} S_i \cap \bigcup_{j \in J} T_j \right| \geqslant \frac{m}{n}(|I| + |J| - n) \geqslant |I| + |J| - n.$$

Hence, by Theorem 6, $\mathscr{S}$ and $\mathscr{T}$ have a common transversal.

Note that the case $k' = l' = 1$ in Theorem 7 is equivalent to Corollary 3. Moreover, the König Theorem mentioned below Corollary 3 (and equivalent to it) is often stated in this form (e.g., see [9, Proof of Theorem 5.1]). So we see that Theorem 7 is, in fact, a generalization of that result of König.

Theorem 7 does not generalize in the obvious way to more than two families of sets. For example, the families

$$\mathscr{R} = (\{1, 2\}, \{3, 4\}),$$
$$\mathscr{S} = (\{1, 3\}, \{2, 4\}),$$
$$\mathscr{T} = (\{1, 4\}, \{2, 3\})$$

have no common transversal despite their regularity properties. In fact, essentially nothing is known about common transversals of more than two families of sets. The difficulty here is closely related to the problem of intersecting more than two matroids. See, e.g., [2].

In terms of 0-1 matrices, Theorem 7 says:

THEOREM 7′. *Given two $n \times m$ 0-1 matrices ($m \geqslant n$) each with constant row sums (> 0) and constant column sums (>0), there exists $N \subset \{1,..., m\}$ with $| N | = n$ such that each matrix contains a set of n independent* 1'*s taken from the columns indexed by N.*

Theorem 7 can be strengthened slightly in the way that Corollary 5 strengthened Corollary 3:

COROLLARY 8. *Let $\mathscr{S} = (S_1 ,..., S_n)$ and $\mathscr{T} = (T_1 ,..., T_n)$ be finite families of subsets of a finite set $E = \{e_1 ,..., e_m\}$. If*

$$(n + 1) \cdot \min_{1 \leqslant i \leqslant n} | S_i | \geqslant (m + 1) \cdot \max_{1 \leqslant j \leqslant m} |\{i \mid e_j \in S_i\}|$$

and

$$(n + 1) \cdot \min_{1 \leqslant i \leqslant n} | T_i | \geqslant (m + 1) \cdot \max_{1 \leqslant j \leqslant m} |\{i \mid e_j \in T_i\}|,$$

then $\mathscr{S}$ and $\mathscr{T}$ have a common transversal.

Proof. Let

$$k = \min_{1 \leqslant i \leqslant n} | S_i |, \qquad k' = \max_{1 \leqslant j \leqslant m} |\{i \mid e_j \in S_i\}|,$$
$$l = \min_{1 \leqslant i \leqslant n} | T_i |, \qquad l' = \max_{1 \leqslant j \leqslant m} |\{i \mid e_j \in T_i\}|.$$

(Note that

$$k'm \geqslant \sum_{j=1}^{m} |\{i \mid e_j \in S_i\} = \sum_{i=1}^{n} | S_i | \geqslant kn$$

and hence

$$\frac{m}{n} \geqslant \frac{k}{k'} \geqslant \frac{m + 1}{n + 1}$$

which implies that $m \geqslant n$, as is obviously necessary.)

Now we must show that

$$\left| \bigcup_{i \in I} S_i \cap \bigcup_{j \in J} T_j \right| \geqslant |I| + |J| - n$$

for all $I, J \subseteq \{1,..., n\}$. Since this clearly holds for $|I| + |J| \leqslant n$, let us take $|I| + |J| = n + 1 + r$ $(0 \leqslant r \leqslant n - 1)$. Then we have

$$\left| \bigcup_{i \in I} S_i \cap \bigcup_{j \in J} T_j \right| \geqslant \frac{k \cdot |I|}{k'} + \frac{l \cdot |J|}{l'} - m$$

$$\geqslant \frac{m+1}{n+1} \cdot |I| + \frac{m+1}{n+1} \cdot |J| - m.$$

But

$$\frac{m+1}{n+1} \cdot (|I| + |J|) - m = \frac{m+1}{n+1} \cdot (n + 1 + r) - m$$

$$= \frac{m+1}{n+1} \cdot r + 1$$

and

$$\frac{m+1}{n+1} \cdot r + 1 \geqslant r + 1 = |I| + |J| - n,$$

so

$$\left| \bigcup_{i \in I} S_i \cap \bigcup_{j \in J} T_j \right| \geqslant |I| + |J| - n,$$

and hence, by Theorem 6, we are done.

Note that, since

$$n \cdot \min_{1 \leqslant i \leqslant n} |S_i| \leqslant m \cdot \max_{1 \leqslant j \leqslant m} |\{i \mid e_j \in S_i\}|$$

always (and similarly for $\mathscr{T}$), the conditions in the above corollary occur rather infrequently. For example, for $m = 100$ and $n = 20$ the only possible situations which satisfy such a condition are

k'	k	k'	k	k'	k
1	5	8	39, 40	15	73, 74, 75
2	10	9	44, 45	16	77, 78, 79, 80
3	15	10	49, 50	17	82, 83, 84, 85
4	20	11	53, 54, 55	18	87, 88, 89, 90
5	25	12	58, 59, 60	19	92, 93, 94, 95
6	29, 30	13	63, 64, 65	20	97, 98, 99, 100
7	34, 35	14	68, 69, 70		

where

$$k' = \max_{1 \leqslant j \leqslant 100} |\{i \mid e_j \in S_i\}|$$

and

$$k = \min_{1 \leqslant i \leqslant 20} | S_i |.$$

4. Infinite Extensions

Folkman and Fulkerson [5, Remark 11] have given an infinite generalization of Theorem 6:

THEOREM 9. *Let $\mathscr{S} = (S_i)_{i \in I}$ and $\mathscr{T} = (T_j)_{j \in J}$ be two families of finite subsets of a set E, such that $|\{i \mid e \in S_i\}| < \infty$ and $|\{i \mid e \in T_i\}| < \infty$ for all $e \in E$. Then $\mathscr{S}$ and $\mathscr{T}$ have a common transversal if and only if*

$$\left| \bigcup_{i \in I - I'} S_i \cap \bigcup_{j \in J'} T_j \right| \geqslant | J' | - | I' |$$

and

$$\left| \bigcup_{i \in I'} S_i \cap \bigcup_{j \in J - J'} T_j \right| \geqslant | I' | - | J' |$$

for all finite $I' \subseteq I$ and finite $J' \subseteq J$.

We now extend Theorem 7 to this case:

THEOREM 10. *Let $\mathscr{S} = (S_i)_{i \in I}$ and $\mathscr{T} = (T_j)_{j \in J}$ be two families of finite subsets of a set E. If $\mathscr{S}$ is (k, k')-regular and $\mathscr{T}$ is (l, l')-regular with $k/k' = (l/l') \geqslant 1$, then $\mathscr{S}$ and $\mathscr{T}$ have a common transversal.*

(*Note*: in the finite case, $k/k' = l/l'$ automatically and $\geqslant 1$ follows from the necessary assumption that $| E | \geqslant n$.)

Proof. Let I' and J' be finite subsets of I and J, respectively. Then

$$\bigcup_{i \in I - I'} S_i = E - R,$$

where

$$R = \{e \in E \mid e \in S_i \text{ for } k' \text{ distinct } i \in I'\}.$$

Now clearly

$$| R | \leqslant \frac{k \cdot | I' |}{k'}$$

and also

$$\left| \bigcup_{j \in J'} T_j \right| \geqslant \frac{l \cdot |J'|}{l'}.$$

So

$$\begin{aligned}\left| \bigcup_{i \in I - I'} S_i \cap \bigcup_{j \in J} T_j \right| &= \left| (E - R) \cap \bigcup_{j \in J'} T_j \right| \\ &= \left| \bigcup_{j \in J'} T_j \right| - \left| \bigcup_{j \in J'} T_j \cap R \right| \\ &\geqslant \left| \bigcup_{j \in J'} T_j \right| - |R| \\ &\geqslant \frac{l \cdot |J'|}{l'} - \frac{k \cdot |I'|}{k'} \\ &\geqslant |J'| - |I'|.\end{aligned}$$

Similarly, we have

$$\left| \bigcup_{i \in I'} S_i \cap \bigcup_{j \in J - J'} T_j \right| \geqslant |I'| - |J'|.$$

So, by Theorem 9, $\mathscr{S}$ and $\mathscr{T}$ have a common transversal.

The following series of examples gives some insight into the nature of the "extra" conditions in Theorem 10; i.e., the conditions

(i) $\dfrac{k}{k'} = \dfrac{l}{l'}$

and

(ii) $\dfrac{k}{k'}, \dfrac{l}{l'} \geqslant 1.$

1. (i) is not necessary for the existence of a common transversal; e.g., let

$$A = \{a_1, a_2, \ldots\} = \{1, 4, 7, 10, \ldots\}$$

and

$$B = \{b_1, b_2, \ldots\} = Z^+ - A = \{2, 3, 5, 6, 8, 9, \ldots\}$$

and consider the families

$$\mathscr{S} = (\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}, \{10, 11, 12\}, \ldots)$$

and

$$\begin{aligned}\mathscr{T} &= (\{a_{2i-1}, a_{2i}, b_i\}, \{a_{2i-1}, a_{2i}, b_i\}), \qquad i = 1, 2, \ldots \\ &= (\{1, 4, 2\}, \{1, 4, 2\}, \{7, 10, 3\}, \{7, 10, 3\}, \{13, 16, 5\}, \{13, 16, 5\}, \ldots).\end{aligned}$$

These families have $k/k' = 3 \neq 3/2 = l/l'$, but the set A is clearly a common transversal.

2. (i) is essential to the theorem (i.e., (ii) alone is not sufficient); e.g., consider the families

$$\mathscr{S} = (\{1, 2\}, \{1, 2\}, \{3, 4\}, \{3, 4\}, \{5, 6\}, \{5, 6\},...)$$

and

$$\mathscr{T} = (\{1, 2, 3\}, \{1, 2, 3\}, \{4, 5, 6\}, \{4, 5, 6\},...).$$

These families have k/k' and $l/l' \geqslant 1$, but they clearly have no common transversal.

3. (ii) is not necessary for the existence of a common transversal; e.g., let

$$\begin{aligned}\mathscr{R}_k = (&\{2^{k-1} + 1, 2^k + 1\}, \{2^{k-1} + 1, 2^k + 2\},\\ &\{2^{k-1} + 2, 2^k + 3\}, \{2^{k-1} + 2, 2^k + 4\},\\ &\{2^{k-1} + 3, 2^k + 5\}, \{2^{k-1} + 3, 2^k + 6\},\\ &\vdots\\ &\{2^k, 2^{k+1} - 1\}, \{2^k, 2^{k+1}\})\end{aligned}$$

and consider the families

$$\begin{aligned}\mathscr{S} = \mathscr{T} &= (\{1, 2\}, \{1, 2\}, \{1, 3\}, \{2, 4\}) \cup \mathscr{R}_2 \cup \mathscr{R}_3 \cup \mathscr{R}_4 \cup \cdots\\ &= (\{1, 2\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 5\}, \{3, 6\}, \{4, 7\}, \{4, 8\}, \{5, 9\},\\ &\quad \{5, 10\}, \{6, 11\}, \{6, 12\}, \{7, 13\}, \{7, 14\}, \{8, 15\}, \{8, 16\},...).\end{aligned}$$

Then $k/k' = l/l' = 2/3 < 1$, but $Z^+ = \{1, 2,...\}$ is a common transversal.

4. (ii) is essential to the theorem; e.g., consider the families

$$\mathscr{S} = \mathscr{T} = (\{1\}, \{1\}, \{2\}, \{2\}, \{3\}, \{3\},...).$$

Here $k/k' = l/l'\ (= 1/2)$, but there is clearly no common transversal. However, this example avoids the issue somewhat, because here neither $\mathscr{S}$ nor $\mathscr{T}$ has a transversal to begin with. A more reasonable question is whether or not (i) is sufficient to force a common transversal when both $\mathscr{S}$ and $\mathscr{T}$ have transversals. This is certainly the case when $k/k'\ (=l/l') \geqslant 1$ (by Theorem 10), so we need only investigate families with $k/k' < 1$. Note that in the finite case such a family could not have a transversal since $k/k' = m/n$. However, we have already seen such an infinite family with a transversal (Example 3 above).

The following theorem resolves our question in the affirmative:

THEOREM 11. *Let $\mathscr{S} = (S_i)_{i \in I}$ be an infinite family of finite subsets of a set E. If $\mathscr{S}$ is (k, k')-regular with $k \leqslant k'$ and if $\mathscr{S}$ has a transversal, then E itself is a transversal for $\mathscr{S}$.*

Proof. We make use of the following theorem of Hoffman-Kuhn-Rado (see [11, Theorem 6.6.3]):

LEMMA. *Let $\mathscr{S}$ be an infinite family of finite subsets of a set E, and let F be a subset (finite or infinite) of E such that $|\{i \mid f \in S_i\}| < \infty$ for all $f \in F$. Then $\mathscr{S}$ has a transversal which contains F if and only if both Hall's condition and the following condition hold: for any finite set $F' \subseteq F$, the number of sets S_i that meet F' is at least $|F'|$.*

Now in our case, each element of E occurs in $k' < \infty$ sets S_i, so the hypothesis of the lemma is satisfied. We are given that $\mathscr{S}$ has a transversal, so Hall's condition holds (by Theorem 2).

Now let A be a finite subset of E. Each element of A occurs in k' sets, and no more than k of these $k' \cdot |A|$ occurrences can be in the same set, so A hits at least $(k' \cdot |A|)/k$ sets S_i; i.e.,

$$|\{i \mid A \cap S_i \neq \varnothing\}| \geqslant \frac{k' \cdot |A|}{k} \geqslant |A|$$

and hence the other condition of the lemma is satisfied. So E is contained in a transversal for $\mathscr{S}$, which is to say E *is* a transversal for $\mathscr{S}$.

Hence, with regard to the question raised after Example 4 above, we have:

COROLLARY 12. *Let $\mathscr{S} = (S_i)_{i \in I}$ and $\mathscr{T} = (T_j)_{j \in J}$ be infinite families of finite subsets of a set E, each with a transversal. If $\mathscr{S}$ is (k, k')-regular and $\mathscr{T}$ is (l, l')-regular with $k/k' = l/l'$, then $\mathscr{S}$ and $\mathscr{T}$ have a common transversal.*

Proof. If $k/k'\ (= l/l') > 1$, the result follows from Theorem 10. If $k/k'\ (= l/l') \leqslant 1$, E itself is a common transversal by Theorem 11.

From Theorem 11 we also get immediately the following result, which we state explicitly only because so little is known about three or more families:

COROLLARY 13. *Let $\mathscr{S}_p = (S_{pi})_{i \in I}$, $p \in P$, be infinite families of finite subsets of a set E, each with a transversal. If $\mathscr{S}_p$ is (k_p, k_p')-regular with $k_p \leqslant k_p'$ for all $p \in P$, then all the families have a common transversal (namely, E itself).*

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